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A Classification of Fokker–Planck Models and the Small and Large Noise Asymptotics

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The n-variable Fokker–Planck equation can be written in an equivalent form as a system of $n+1$ first-order equations by introducing as auxiliary variables the components of the drift velocity R_i. The stationary state defines a stationary R uniquely, which allows an intrinsic classification of the stationary states in terms of the properties of R, without reference to detailed balance. This representation is very appropriate for the study of questions such as the existence of stationary states and their small and large noise asymptotics, as well as for the construction of models having some specified behavior. R provides also a classification of the dynamics, which corresponds to the hermiticity properties of the associated eigenvalue problem.

KEY WORDS: Fokker–Planck equation; drift velocity; detailed balance; nonequilibrium.

1. INTRODUCTION

The Fokker–Planck equation can be used as a model for a wide variety of fluctuating macroscopic systems.[1–4] Detailed balance is a property of the stationary state defined with respect to a time-reversal transformation. It presupposes the assignment of specific time-reversal signatures to the variables of the system, which reflect the nature of the underlying physical system. The concept of detailed balance is useful in many contexts. If a system is in detailed balance, the stationary state can be obtained directly by quadratures. It is a covariant property, i.e., independent of the choice of coordinates. However, as a classification criterion it has some problems: since it is defined with respect to some given time-reversal signatures, it is bounded to an interpretation of the macroscopic variables; a stationary

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state can be in detailed balance with respect to one interpretation and not for another one, although the equation is the same in both cases. Detailed balance was known to be equivalent to the potential conditions of Graham and Haken.[5,6] R. Graham showed later[7,8] that they are satisfied for all Fokker–Planck models, and every stationary state is formally in detailed balance with respect to some time reversal transformation. In fact the potential conditions are equivalent to the Fokker–Planck equation. Thus in many instances it is useful to have a more precise criterion. We consider an alternative classification criterion, based on the properties of the stationary drift velocity R^μ, which is uniquely defined by the stationary state. This classification is independent of the coordinates and of the interpretation of the variables (i.e., of time reversal). It coincides in one case with the one based on detailed balance. The study of the properties of R^μ gives also useful information on the structure of the Fokker–Planck models, for the stationary state and for the dynamics. It allows, e.g., to identify a new class of models for which the stationary state can be found explicitly.

The paper is organized as follows: after giving some notation and introducing the covariant formulation of the Fokker–Planck equation, we present in Section 2 three equivalent representations of the Fokker–Planck equation, on which we base the rest of the discussion. In Section 3 we discuss the dynamical-matrix formalism and observe that it includes all models having a stationary state. In Section 4 we give an intrinsic classification of stationary states, based on the behavior of the drift velocity R^μ, and discuss the properties of the different cases. We find a method to determine to which class a given model belongs and how to construct models having specified properties. We treat as an example a family of models that undergo a Hopf bifurcation into a limit cycle and contains the stochastic van der Pol oscillator as a special case. We conclude, e.g., that for systems in equilibrium one must require $\nabla_\mu R^\mu = 0$. In Section 5 we give some results on the small and large noise asymptotics in relation with the foregoing classification. In Section 6 we discuss how the properties of R^μ relate to the dynamics through the hermiticity properties of the associated Fokker–Planck operator.

We will use the following notation: We consider the Fokker–Planck equation (FP) in n variables $q \equiv (q^1,..., q^n)$ in a simply connected domain $\Omega \subset \mathbb{R}^n$

$$\partial_t P_t(q) = -\partial_\mu [K^\mu(q) P_t(q)] + \tfrac{1}{2}\partial_\mu \partial_\nu [\eta \hat{Q}^{\mu\nu}(q) P_t(q)] \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $\partial_\mu = \partial/\partial q^\mu$. The summation convention applies for repeated indices. We will also use $\partial\phi \equiv (\partial_1 \phi,..., \partial_n \phi)$. We will distinguish the time-dependent quantities (e.g., ϕ_t) from the corresponding stationary ones (e.g., ϕ) by a subscript t. The positive parameter η appearing in the diffusion tensor $Q^{\mu\nu} = \eta \hat{Q}^{\mu\nu}$ (positive, symmetric) measures the strength of the fluctuations. In equilibrium it can be identified with the Boltzmann constant k_B. For simplicity we assume that $Q^{\mu\nu}$ has an inverse $Q_{\mu\nu}$. We choose vanishing boundary conditions for P_t, its derivatives, and the probability current. Since $P_t(q) \geqslant 0$ we can write it as

$$P_t(q) = e^{-\phi_t(q)} \tag{1.2}$$

The vanishing boundary conditions for P_t correspond to ϕ_t becoming infinite at the boundaries.

Since we are interested in properties that do not depend on the choice of coordinates, we formulate the FP equation in a covariant form.[9] Introducing the scalar

$$S_t \equiv |Q|^{1/2} P_t \equiv e^{-\bar{\phi}_t} = e^{-\phi_t + (1/2)\ln|Q|} \tag{1.3}$$

where $|Q| = \det(Q^{\mu\nu})$; the contravariant drift

$$h^\mu \equiv K^\mu - \frac{1}{2} |Q|^{1/2} \partial_\alpha \frac{Q^{\alpha\mu}}{|Q|^{1/2}} \tag{1.4}$$

and the covariant derivative (notation ∇_μ) the covariant FP equation[9] is

$$\partial_t S_t = -\nabla_\mu [h^\mu S_t] + \tfrac{1}{2} Q^{\mu\nu} \nabla_\mu \nabla_\nu S_t \tag{1.5}$$

We remark that since $\nabla_\alpha Q^{\mu\nu} = 0$, Eq. (1.5) is of the same form as (1.1) in the case $\partial_\alpha Q^{\mu\nu} = 0$ (i.e., q-independent diffusion), if one makes the substitutions $P_t \to S_t$, $\partial_\mu \to \nabla_\mu$, $K^\mu \to h^\mu$. Therefore many results obtained for (1.1) in the case $\partial_\alpha Q^{\mu\nu} = 0$ by formal procedures, can be generalized to the case of q-dependent $Q^{\mu\nu}$ by repeating the analogous procedures on Eq. (1.5) (one must take into account that the covariant derivatives ∇_μ, ∇_ν do not commute when applied on vectors). We also remark that if in a coordinate system $Q^{\mu\nu}$ is constant, Eq. (1.5) in this coordinates reduces to Eq. (1.1).

We will consider time-reversal transformations τ acting on the variables q and on external parameters $\Lambda = (\Lambda^1, ..., \Lambda^p)$[9]

$$\begin{aligned} \tau\colon q^\nu &\to \tilde{q}^\nu = t^\nu_\mu q^\mu \\ \Lambda^i &\to \tilde{\Lambda}^i = s^i_j \Lambda^j \end{aligned} \tag{1.6}$$

with

$$t^\nu_\mu t^\mu_\alpha = \delta^\nu_\alpha, \qquad s^i_j s^j_k = \delta^i_k \tag{1.7}$$

The tensors t_μ^ν and s_j^i can be diagonalized by choosing appropriate coordinates. They have eigenvalues $\varepsilon^\nu = \pm 1$ and $\sigma^i = \pm 1$, respectively, which are the signatures of the corresponding coordinates and parameters with respect to τ.

Detailed balance with respect to τ is defined by the condition on the two-time probability density

$$W_2(q', t'; q, t; \Lambda) = W_2(\tilde{q}, t'; \tilde{q}', t; \tilde{\Lambda}) \tag{1.8}$$

and is a coordinate independent property.

We distinguish three types of detailed balance:

db(1): All the ε^ν and σ^i are $+1$. The process is balanced locally.

db(2): Some of the ε^ν are -1 but all the σ^i are $+1$. The balance is nonlocal.

db(3): There are some $\sigma^i = -1$. The balance does not reflect a process of the system since it involves changing external parameters.

2. EQUIVALENT FORMS OF THE FOKKER–PLANCK EQUATION

The FP Eq. (1.5) is a linear second-order partial differential equation, and second-order nonlinear as an equation for $\bar{\phi}$. By introducing n auxiliary functions $R_t^\nu(q)$ we can write it as a system of $n+1$ first-order equations (2.1) that will be the basis of all further analysis. We give also two other equivalent systems that will be used later.

The three following systems are equivalent to the FP Eq. (1.5):

(A)
$$R_t^\mu = h^\mu + \tfrac{1}{2} Q^{\mu\nu} \nabla_\nu \bar{\phi}_t \tag{2.1a}$$
$$\partial_t \bar{\phi}_t + R_t^\mu \nabla_\mu \bar{\phi}_t = \nabla_\mu R_t^\mu \tag{2.1b}$$

(B)
$$R_t^\mu = h^\mu + \tfrac{1}{2} Q^{\mu\nu} \nabla_\nu \bar{\phi}_t \tag{2.2a}$$
$$\partial_t \bar{\phi}_t + H(q, \nabla \bar{\phi}_t) = \nabla_\mu R_t^\mu \tag{2.2b}$$

with
$$H(q, \nabla \bar{\phi}_t) = \tfrac{1}{2} Q^{\mu\nu} \nabla_\mu \bar{\phi}_t \nabla_\nu \bar{\phi}_t + h^\mu \nabla_\mu \bar{\phi}_t \tag{2.3}$$

(C)
$$R_t^\mu = h^\mu + \tfrac{1}{2} Q^{\mu\nu} \nabla_\nu \bar{\phi}_t \tag{2.4a}$$
$$\partial_t \bar{\phi}_t + R_t^\mu \nabla_\mu \bar{\phi}_t = \tfrac{1}{2} Q^{\mu\nu} \nabla_\mu \nabla_\nu \bar{\phi}_t + \nabla_\nu h^\nu \tag{2.4b}$$

To show this we write the FP Eq. (1.5) as a continuity equation

$$\partial_t S_t = -\nabla_\mu J_t^\mu \tag{2.5}$$

with

$$J_t^\mu = h^\mu S_t - \tfrac{1}{2} Q^{\mu\nu} \nabla_\nu S_t$$
$$= (h\mu^\mu + \tfrac{1}{2} Q^{\mu\nu} \nabla_\nu \bar{\phi}_t)\, S_t \equiv R_t^\mu S_t \tag{2.6}$$

This defines the drift velocity R_t^μ and coincides with Eq. (2.1a). Inserted into (2.5) it gives

$$\partial_t \bar{\phi}_t = -R_t^\mu \nabla_\mu \bar{\phi}_t + \nabla_\mu R_t^\mu \tag{2.7}$$

which coincides with (2.1b).

We remark that in any coordinate system, R^μ is given by

$$R_t^\mu = K^\mu + \tfrac{1}{2} Q^{\mu\nu} \partial_\nu \phi_t - \tfrac{1}{2} \partial_\nu Q^{\mu\nu} \tag{2.8}$$

which, as it is easily verified, is a covariant expression; i.e., the drift velocity defined in any coordinate system is identical with its covariant counterpart. We also remark that R_t^μ is uniquely defined for all t, as follows from Eq. (2.8) and the uniqueness of ϕ_t for given initial and boundary conditions.

For the equivalence with the two other systems we notice that (2.2b) is obtained by inserting (2.1a) into the left side of (2.1b), while insertion into the right side gives (2.4b). As we will see later, the forms (B) and (C) are useful for the study of the small and large noise asymptotics, respectively. We notice from (2.8) that the knowledge of ϕ_t gives R_t^μ immediately, and ϕ_t can be obtained from R_t^μ by integration.

The system (2.1) corresponds to the time-dependent generalization of the potential conditions of Graham and Haken.[5,6] Its equivalence to the FP equation is the origin of the observation that every stationary state is in detailed balance db(3) with respect to some time-reversal transformation.[7]

3. STATIONARY STATES

One knows[10,11] that, under fairly general conditions, the time-dependent solutions P_t of the FP Eq. (1.1) approach a stationary state P_∞ if and only if the stationary FP equation has a normalized solution P. In that case $P_\infty \equiv P$ is unique. For the stationary FP equation we can formulate equivalent systems of equations corresponding to (2.1)–(2.4), just by substituting $\bar{\phi}_t$ and R_t^μ by $\bar{\phi}$ and R^μ and dropping the terms $\partial_t \bar{\phi}_t$; e.g.,

$$R^\mu = h^\mu + \tfrac{1}{2} Q^{\mu\nu} \nabla_\nu \bar{\phi} \tag{3.1a}$$

$$R^\mu \nabla_\mu \bar{\phi} = \nabla_\mu R^\mu \tag{3.1b}$$

We want to discuss some properties that characterize the h^ν for which the FP Eq. (1.5) has a stationary state. We start by reinterpreting the dynamical-matrix formalism developed by C. P. Enz.[12] Let $Q^{\mu\nu}(q)$ and $\bar{\phi}(q)$ be given, satisfying the necessary regularity and boundary conditions. We want to determine all the covariant drifts h^ν for which $\bar{\phi}$ is the stationary state. We define

$$\bar{D}^\mu = -\tfrac{1}{2}Q^{\mu\nu}\nabla_\nu\bar{\phi} = h^\mu - R^\mu \tag{3.2}$$

and determine the general solution $R^\mu(q)$ of Eq. (3.1b), which is given by[25]

$$R^\mu = -\tfrac{1}{2}d^{\mu\nu}\nabla_\nu\bar{\phi} + \tfrac{1}{2}\nabla_\nu d^{\mu\nu} \tag{3.3}$$

where $d^{\mu\nu}(q) = -d^{\nu\mu}(q)$ is any antisymmetric $n\times n$ matrix with C^2 coefficients. Thus h^μ will be of the form

$$h^\mu = \bar{D}^\mu + R^\mu = B^{\mu\nu}\nabla_\nu\bar{\phi} - \nabla_\nu B^{\mu\nu} \tag{3.4}$$

where we have defined the dynamical matrix

$$B^{\mu\nu} = -\tfrac{1}{2}(Q^{\mu\nu} + d^{\mu\nu}) \tag{3.5}$$

This implies, as mentioned, e.g., in Ref. 9, that a FP Eq. (1.5) defined by some h^μ and $Q^{\mu\nu}$ has a stationary state if and only if it can be written in the two following equivalent forms:

$$\text{(i)}\quad \partial_t S_t = -\nabla_\mu[(B^{\mu\nu}\nabla_\nu\bar{\phi} - \nabla_\nu B^{\mu\nu})\,S_t] + \tfrac{1}{2}Q^{\mu\nu}\nabla_\mu\nabla_\nu S_t \tag{3.6}$$

$$\text{(ii)}\quad \partial_t S_t = -\nabla_\mu\left[B^{\mu\nu}\left(-\frac{\nabla_\nu S}{S}S_t + \nabla_\nu S_t\right)\right] \tag{3.7}$$

Equation (3.7) is the form of FP equation used by Green, Grabert, and Graham.[13] Its equivalence with (3.6) is obtained by simple algebraic manipulation. This shows that this formalism[13] as well as the dynamical-matrix approach starts *ab initio* with the most general form of Fokker–Planck equation having a stationary state. Each particular model is characterized by a choice of $\bar{\phi}$, $Q^{\mu\nu}$, and $d^{\mu\nu}$. This gives a very transparent method to construct models having some specified properties. $\bar{\phi}$ describes the stationary probability density, $Q^{\mu\nu}$ the diffusion, and $d^{\mu\nu}$ the convection properties. In Section 6 we will discuss the relation of this choices with the dynamics. Some general sufficient conditions on h^μ and $Q^{\mu\nu}$ for the system to have a stationary state are given in Ref. 14.

4. CLASSIFICATION OF STATIONARY STATES

We consider systems that have a stationary state. For given $Q^{\mu\nu}$, h^μ, the stationary state is uniquely characterized by $\bar{\phi}$ or R^μ. Since R^μ is contravariant and defined independently of the interpretation given to the variables q, and in particular of any time-reversal transformation, we can use it for an intrinsic classification of stationary states. We will distinguish three cases:

$$\text{(I)}\quad R^\mu = 0 \tag{4.1}$$

$$\text{(II)}\quad \nabla_\mu R^\mu = 0 \tag{4.2}$$

$$\text{(III)}\quad \nabla_\mu R^\mu \neq 0 \tag{4.3}$$

This classification is independent of the choice of coordinates. It reflects both the mathematical structure of the equations and the physical nature of the involved processes.

Case I. (a) $R^\mu = 0$, the state is purely diffusive; the fluctuations relax without performing any collective motion.

(b) The stationary FP equation reduces to

$$h^\mu = -\tfrac{1}{2} Q^{\mu\nu} \nabla_\nu \bar{\phi} \tag{4.4}$$

and thus $\bar{\phi}$ can be determined by quadrature:

$$\bar{\phi} = -2 \int_{q_0}^{q} Q_{\lambda\mu} h^\mu \, dl^\lambda \tag{4.5}$$

(c) Conversely, if the integral (4.5) is well defined and independent of the path (and the corresponding P is normalizable), it implies $R^\mu = 0$, since it satisfies (3.1) and we have uniqueness of the solution. Thus it can be tested *a priori*: $R^\mu = 0$ if

$$\partial_\beta [Q_{\lambda\mu} h^\mu] = \partial_\lambda [Q_{\beta\mu} h^\mu] \tag{4.6}$$

Here we use the assumption that $\Omega \subset \mathbb{R}^n$ is simply connected, so that Poincaré's lemma applies. For $n = 1$ the integral (4.5) is trivially independent of the path and therefore we always have $R^\mu = 0$.

(d) $R^\mu = 0$ if and only if the system is in db(1), i.e., detailed balance with all signatures (1.7) positive. This is the only case in which this classification coincides with detailed balance. One can see this property as follows: we define in general

$$^{\tau}R^\nu(q, \Lambda) = \tfrac{1}{2}[h^\nu(q, \Lambda) - t^\nu_\alpha h^\alpha(\tilde{q}, \tilde{\Lambda})] \tag{4.7}$$

where τ is the time-reversal transformation (1.9). Detailed balance with respect to τ is equivalent to the potential conditions[5,6] of R. Graham and H. Haken. By comparison with (3.1) and from the uniqueness of R^μ one sees that detailed balance is equivalent to

$$\text{(i)} \quad {}^{\tau}R^\mu = R^\mu \tag{4.8}$$

$$\text{(ii)} \quad Q^{\mu\nu}(q) = t^\mu_\alpha t^\nu_\beta Q^{\alpha\beta}(\tilde{q}) \tag{4.9}$$

Now, db(1) implies ${}^{\tau}R^\mu = 0$ and therefore $R^\mu = 0$. Conversely, if $R^\mu = 0$ Eq. (3.1) is identical to the potential conditions corresponding to db(1).

Case II. (a) The condition $\nabla_\mu R^\mu = 0$ is equivalent to $\nabla_\mu \bar{\phi} R^\mu = 0$. $R^\mu = 0$ is a special case and the following properties (b)–(h) also apply to it. If $R^\mu \neq 0$, $\nabla_\mu R^\mu = 0 = \nabla_\mu \bar{\phi} R^\mu$ the relaxation has a diffusive and a convective part, which is source free and happens along equipotential hypersurfaces.

We remark that there always exist coordinate systems in which the divergence coincides with the covariant divergence, i.e., $\nabla_\mu V^\mu = \partial_\mu V^\mu$ for any vector V^μ. This coordinates can even be constructed explicitly.[25] Thus if $\nabla_\mu R^\mu = 0$ there are coordinates in which $\partial_\mu R^\mu = 0$, and we could make all the following analysis in one of this coordinate systems. However, the converse is not true: it is possible that in some coordinate system $\partial_\mu R^\mu = 0$ although $\nabla_\mu R^\mu \neq 0$. Some of the following considerations can also be applied to this case [like, e.g., point (c)].

(b) The time-independent FP systems (2.2) and (2.4) simplify, respectively, to

$$R^\mu = h^\mu + \tfrac{1}{2} Q^{\mu\nu} \nabla_\nu \bar{\phi} \tag{4.10a}$$

$$H(q, \nabla\bar{\phi}) = 0 \tag{4.10b}$$

$$H(q, \nabla\bar{\phi}) = \tfrac{1}{2} Q^{\mu\nu} \nabla_\mu \bar{\phi} \nabla_\nu \bar{\phi} + h^\nu \nabla_\nu \bar{\phi} \tag{4.11}$$

and

$$R^\mu = h^\mu + \tfrac{1}{2} Q^{\mu\nu} \nabla_\nu \bar{\phi} \tag{4.12a}$$

$$Q^{\mu\nu} \nabla_\mu \nabla_\nu \bar{\phi} = -2 \nabla_\nu h^\nu \tag{4.12b}$$

In both representations, the last equation (b) decouples from the rest of the system. Either (4.10b) or (4.12b), which are much simpler than the full FP equation, can be used to determine $\bar{\phi}$. (4.10b) has the form of a time-independent Hamilton–Jacobi equation (HJ), which can be reduced to a system of ordinary differential equations by the method of characteristics.

(4.12b) has the form of a generalized Poisson equation, which can be treated by Green's function methods.[15] If $\nabla_\mu R^\mu = 0$ Eqs. (4.10b) and (4.12b) give another equivalent representation of the FP equation: as an equation for $\bar{\phi}$ the stationary FP Eq. (1.5) can be written

$$\tfrac{1}{2}Q^{\mu\nu}\nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi} + h^\nu\nabla_\nu\bar{\phi} = \tfrac{1}{2}Q^{\mu\nu}\nabla_\mu\nabla_\nu\bar{\phi} + \nabla_\nu h^\nu \tag{4.13}$$

The left side corresponds to the HJ equation and the right side to the Poisson equation. Thus in the case $\nabla_\mu R^\mu = 0$ the FP equation is equivalent to

$$H(q, \nabla\bar{\phi}) = 0 \tag{4.14a}$$

$$Q^{\mu\nu}\nabla_\mu\nabla_\nu\bar{\phi} = -2\nabla_\nu h^\nu \tag{4.14b}$$

As we will see in Section 5, the HJ equation can be associated with the small-noise limit, and the Poisson equation with the large-noise limit.

(c) We do not have a criterion to determine, for given h^ν, $Q^{\mu\nu}$, *a priori*, if $\nabla_\mu R^\mu = 0$ (i.e., without finding $\bar{\phi}$ first). However, since the Poisson equation (4.14b) is known to have regular solutions under general conditions[15] and it can be solved explicitly in many instances, we can proceed as follows: after checking that $R^\mu \neq 0$ we try to determine the general solution of (4.14b) satisfying the boundary conditions. For example, if h^ν and $Q^{\mu\nu}$ are polynomial and $\Omega = \mathbb{R}^n$, (4.14b) can be solved by a polynomial ansatz, whose coefficients are determined by linear algebraic equations. If this general solution contains a special solution that also satisfies (4.14a) (which can be checked by insertion) it gives us the stationary state, and it will be of type (II). If this is not the case or the boundary conditions cannot be satisfied, either the system will not have a stationary state or it will be of type (III). This procedure can also be performed by starting with the Hamilton–Jacobi equation (4.14a) and then checking if (4.14b) is also satisfied. However, it is generally more difficult to find solutions of (4.14a) and if $\nabla_\mu R^\mu \neq 0$, (4.14a) will in general not have a regular solution.[16] We remark that Eq. (4.4) for the class (I) is a special case both of (4.14a) and (4.14b).

(d) We illustrate the foregoing procedure with an example in two variables defined by the Langevin equation

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + (b - x^2 - \alpha y^2)\,y + \xi(t) \end{aligned} \tag{4.15}$$

where $\xi(t)$ is a Gaussian white noise with $\langle \xi(t)\,\xi(t')\rangle = \delta(t - t)$ and α, b parameters, $0 \leqslant \alpha \leqslant 1$, $-\infty < b < \infty$. This model belongs to the class studied in Refs. 17 and 8. For fixed α and $b<0$ it has a single stable attractor which undergoes a Hopf bifurcation at $b=0$ to become a limit cycle for $b>0$. For $\alpha=1$ the limit cycle is a circle with radius b, and for $\alpha<1$ it deforms into a more rectangular shape. For $\alpha=0$ we get the stochastic van der Pol oscillator. We will determine for which α the property $\partial_\mu R^\mu=0$ is satisfied (the noninvertibility of $Q^{\mu\nu}$ does not affect the procedure). The corresponding Poisson equation (4.14b) is

$$\tfrac{1}{2}\partial_y \partial_y \tilde{\phi} = -(b - x^2 - 3\alpha y^2) \tag{4.16}$$

which can be solved by the polynomial ansatz

$$\tilde{\phi} = C_{20}x^2 + C_{02}y^2 + C_{22}x^2y^2 + C_{40}x^4 + C_{04}y^4 \tag{4.17}$$

Comparison of coefficients gives

$$C_{02} = -b, \qquad C_{22} = 1, \qquad C_{04} = \alpha/2 \tag{4.18}$$

The coefficients C_{20}, C_{40} are still free parameters. We now take (4.17) with (4.18) and insert it into the Hamilton–Jacobi equation (4.14a)

$$\tfrac{1}{2}(\partial_y \tilde{\phi})^2 + y\,\partial_x \tilde{\phi} + [-x + (b - x^2 - \alpha y^2)\,y]\,\partial_y \tilde{\phi} = 0 \tag{4.19}$$

Comparison of coefficients gives [apart from relations coinciding with (4.18)]

$$C_{20} = C_{02}, \qquad C_{40} = C_{22}/2 \tag{4.20}$$

which fixes the free parameters. But one also gets (from a term in y^3x) the relation

$$C_{04} = \frac{C_{22}}{2} \tag{4.21}$$

which is in contradiction with (4.18), unless $\alpha = 1$. Therefore the condition $\partial_\mu R^\mu = 0$ is only satisfied if $\alpha = 1$, in which case it is satisfied for all b and one has

$$\phi = -b(x^2 + y^2) + \tfrac{1}{2}(y^2 + x^2)^2 \tag{4.22}$$

and $R' = y$, $R^2 = -x$.

The van der Pol model, e.g., has $\partial_\mu R^\mu \neq 0$. An analogous discussion can be carried through for the more complicated models of Refs. 17 and 8.

(e) There is no connection between the property $\nabla_\mu R^\mu = 0$ and detailed balance db(2), i.e., one can have db(2) and $\nabla_\mu R^\mu \neq 0$ and also $\nabla_\mu R^\mu = 0$ without db(2). It is easy to verify that Ornstein–Uhlenbeck processes always satisfy $\nabla_\mu R^\mu = 0$, but not necessarily db(2).[25]

(f) The construction of models with $\nabla_\mu R^\mu = 0$ for given $\bar{\phi}$, $Q^{\mu\nu}$ can be done as follows: from (3.3) we have

$$\begin{aligned} \nabla_\mu R^\mu &= -\tfrac{1}{2}\nabla_\mu d^{\mu\nu}\,\nabla_\nu\bar{\phi} - \tfrac{1}{2}d^{\mu\nu}\,\nabla_\mu\nabla_\nu\bar{\phi} + \tfrac{1}{2}\nabla_\mu\nabla_\nu d^{\mu\nu} \\ &= -\tfrac{1}{2}\nabla_\mu d^{\mu\nu}\,\nabla_\nu\bar{\phi} \end{aligned} \tag{4.23}$$

where we have used the antisymmetry of $d^{\mu\nu}$. This expression can be made to vanish by an appropriate choice of $d^{\mu\nu}$. In a coordinate system where the divergence and the covariant divergence coincide we can take, e.g., a constant $d^{\mu\nu}$ or one in which all the columns are divergence-free vectors (which are conditions independent of $\bar{\phi}$, $Q^{\mu\nu}$).

(g) A scalar function $\psi(q)$ is defined to be a Q-potential for the deterministic equation

$$\dot{q}^\nu = f^\nu(q) \tag{4.24}$$

if it satisfies the following properties: (i) ψ is globally defined, single-valued C^2, bounded from below; (ii) stationary in the limit sets; (iii) there exists an $r^\mu(q)$ such that

$$f^\mu = -\tfrac{1}{2}Q^{\mu\nu}\,\nabla_\nu\psi + r^\mu \tag{4.25}$$

$$r^\mu\,\nabla_\mu\psi = 0 \tag{4.26}$$

For positive definite $Q^{\mu\nu}$, ψ is then a Liapounov function for (4.24).

If one adds a general white noise perturbation to Eq. (4.24)

$$\dot{q}^\nu = f^\nu(q) + g_i^\nu(q)\,\xi_t^i \tag{4.27}$$

the associated covariant Fokker–Planck equation is of the form (1.5) with

$$Q^{\mu\nu} = g_i^\mu\, g_j^\nu \delta^{ij} \tag{4.28}$$

$$h^\nu = f^\nu + l^\nu \tag{4.29}$$

where

$$l^\nu = \tfrac{1}{2}Q^{\nu\mu} g_i^\lambda \Xi_{k\mu\lambda}\delta^{ik} \tag{4.30}$$

$\Xi_{k\mu\lambda}$ is the holonomity tensor[9]

$$\Xi_{k\mu\lambda} = \nabla_\lambda\, g_{k\mu} - \nabla_\mu\, g_{k\lambda} \tag{4.31}$$

with

$$g_{k\mu} = Q_{\mu\nu} g_k^\nu \tag{4.32}$$

A model is called holonomous if $\Xi_{k\mu\lambda} = 0$, which implies $f^\nu = h^\nu$ (this includes, e.g., all models in which $Q^{\mu\nu}$ is constant in some coordinate system). In this case, if the condition $\nabla_\mu R^\mu = 0$ is satisfied, then $\bar{\phi}$ is a Q-potential for the deterministic equation (4.24).

(h) Thermodynamic equilibrium is described by stationary states that must satisfy some supplementary conditions:

(i) They must be in detailed balance with respect to a physically given time-reversal transformation.

(ii) ϕ is associated with the entropy $\mathscr{s}$ and P is given by

$$P = e^{(1/k_B)\mathscr{s}} \tag{4.33}$$

where k_B is the Boltzmann constant.[13,20]

(iii) ϕ is a Liapounov function.

The comparison of these conditions with property g) and mainly with Eqs. (5.2), (5.3) of the next section implies that systems in equilibrium (if described by a holonomous model) must satisfy $\nabla_\mu R^\mu = 0$. More precisely, even in this case the noncovariant probability P will be of the form [see (1.3), (5.2)]

$$P = |Q|^{-1/2} e^{-(1/\eta)\varphi} \tag{4.34}$$

i.e., it will be of the form (4.23) only in coordinate systems in which the volume element $|Q|^{1/2}$ is constant. The identification with entropy can only be made in such coordinates.

5. LARGE- AND SMALL-NOISE ASYMPTOTICS

In this section we will discuss the dependence of the stationary state on the strength of the noise η, which was introduced by setting $Q^{\mu\nu} = n\hat{Q}^{\mu\nu}$. We want to study the dependence on η of $\phi(q, \eta)$ and $R^\nu(q, \eta)$. In general h^ν will also depend on η, owing to the noise-induced drift l^ν, according to Eqs. (4.29)–(4.32):

$$h^\nu(q, \eta) = f^\nu(q) + \eta \hat{l}^\nu(q) \tag{5.1}$$

In the holonomous case, h^ν is independent of η and we can prove the following properties:

(i) $\bar{\phi}$ is of the form (unnormalized)

$$\bar{\phi}(q, \eta) = \frac{1}{\eta} \bar{\varphi}(q) \tag{5.2}$$

if and only if

$$\nabla_\mu R^\mu(q, \eta = 1) = 0 \tag{5.3}$$

(ii) $\nabla_\mu R^\mu(q, \eta_1) = 0$ for one η_1 implies $\nabla_\mu R^\mu(q, \eta) = 0$ for all η.

(iii) $R^\mu(q, \eta_1) = 0$ for one η_1 implies $R^\mu(q, \eta) = 0$ for all η. $R^\mu(q, \eta)$ is independent of η if and only if $\nabla_\mu R^\mu(q, \eta) = 0$.

(ii) and (iii) imply that in the holonomous case the classification (4.1)–(4.3) is independent of the strength of the fluctuations η. For a discussion of the nonholonomous case see Ref. 21.

Inserting (5.2) into (3.1a) one sees that R^μ (and therefore $\nabla_\mu R^\mu$) is independent of η. Then (3.1b) implies that $\nabla_\mu \bar{\phi} R^\mu = 0 = \nabla_\mu R^\mu$ for all η.

Conversely, $\nabla_\mu R^\mu(q, \eta = 1) = 0$ implies that $\bar{\phi}(q, \eta = 1)$ satisfies simultaneously (4.14a) and (4.14b):

$$\tfrac{1}{2}\hat{Q}^{\mu\nu} \nabla_\mu \bar{\varphi} \nabla_\nu \bar{\varphi} + h^\mu \nabla_\mu \bar{\varphi} = 0 \tag{5.4a}$$

$$\tfrac{1}{2}\hat{Q}^{\mu\nu} \nabla_\mu \nabla_\nu \bar{\varphi} + \nabla_\mu h^\mu = 0 \tag{5.4b}$$

where we have set $\bar{\varphi}(q) = \bar{\phi}(q, \eta = 1)$. We define

$$\bar{\phi}(q, \eta) = \frac{1}{\eta} \bar{\varphi}(q) \tag{5.5}$$

and insert it into (5.4):

$$\eta(\tfrac{1}{2}\eta\hat{Q}^{\mu\nu} \nabla_\mu \bar{\phi} \nabla_\nu \bar{\phi} + h^\nu \nabla_\nu \bar{\phi}) = 0 \tag{5.6a}$$

$$\tfrac{1}{2}\eta\hat{Q}^{\mu\nu} \nabla_\mu \nabla_\nu \bar{\phi} + \nabla_\nu h^\nu = 0 \tag{5.6b}$$

This implies that $\bar{\phi}$ is a solution of the FP equation (4.13) for all η. (ii) and (iii) follow immediately from the above argument setting

$$\bar{\phi}(q, \eta) = \frac{\eta_1}{\eta} \bar{\phi}(q, \eta_1) \tag{5.7}$$

By inserting (5.2) into the FP equation (4.13)

$$\frac{1}{\eta}\left[\frac{1}{2} \hat{Q}^{\mu\nu} \nabla_\mu \bar{\varphi} \nabla_\nu \bar{\varphi} + h^\mu \nabla_\mu \bar{\varphi}\right] = \frac{1}{2} \hat{Q}^{\mu\nu} \nabla_\mu \nabla_\nu \bar{\varphi} + \nabla_\mu h^\mu \tag{5.8}$$

we see that for holonomous systems with $\nabla_\mu R^\mu = 0$ the stationary FP equation reduces to the Hamilton–Jacobi equation (4.14a) in the small-noise limit ($\eta \to 0$) and to the Poisson equation (4.14b) in the large-noise limit ($\eta \to \infty$). For systems with $\nabla_\mu R^\mu \neq 0$ but small enough, one would expect some similar behavior, or at least asymptotically for small and large noise.

5.1. Small-Noise Asymptotics

We consider models in which $Q^{\mu\nu}$ is constant in some coordinate system (i.e., flat metric, which implies holonomity). Then, under some regularity conditions on h^ν, $\bar{\phi}$ has been shown[22,23] to have for small noise the asymptotic form (Ref. 22, Theorems 6.4.3, 4.4.3, 4.3.1)

$$\bar{\phi}(q, \eta) = \frac{1}{\eta} \bar{\varphi}(q) + \theta(q, \eta) \tag{5.9}$$

(plus normalization) with

$$\lim_{\eta \to 0} \eta\theta(q, \eta) = 0 \tag{5.10}$$

If $\bar{\varphi}(q)$ is differentiable (C^2), $\nabla_\mu \theta$, $\nabla_\mu \nabla_\nu \theta$ satisfy

$$\lim_{\eta \to 0} \eta \nabla_\mu \theta(q, \eta) = 0 \tag{5.11a}$$

$$\lim_{\eta \to 0} \eta \nabla_\mu \nabla_\nu \theta(q, \eta) = 0 \tag{5.11b}$$

and $\bar{\varphi}$ satisfies the Hamilton–Jacobi equation (5.4a). This implies that $\bar{\varphi}$ will be a Q-potential for the deterministic equation (4.24). However, in general $\bar{\varphi}$ will not be differentiable and will not be a solution of (5.4a). Indeed, R. Graham and R. Tél have shown,[16] through the study of integrability properties of the associated Hamiltonian system, that this will be the generic case. We remark that when $\bar{\varphi}$ is differentiable, Eqs. (5.9)–(5.11) lead to finite expressions for the drift velocity R^μ and for $\nabla_\mu R^\mu$:

$$R^\nu(q, \eta \to 0) = h^\nu + \tfrac{1}{2}\hat{Q}^{\mu\nu} \nabla_\mu \bar{\varphi} \tag{5.12}$$

$$\nabla_\nu R^\nu(q, \eta \to 0) = \nabla_\nu h^\nu + \tfrac{1}{2}\hat{Q}^{\mu\nu} \nabla_\mu \nabla_\nu \bar{\varphi} \tag{5.13}$$

In the cases where $\bar{\varphi}$ is not C^2, the limit $\eta \to 0$ is still well behaved[22] and we expect similar expressions, but instead of $\nabla_\mu \bar{\varphi}$, $\nabla_\mu \nabla_\nu \bar{\varphi}$ we have to set functions $\psi_\mu^{(1)} \neq \nabla_\mu \bar{\varphi}$, $\psi_{\mu\nu}^{(2)} \neq \nabla_\mu \nabla_\nu \bar{\varphi}$ defined by the asymptotic expansions

$$\nabla_\mu \bar{\phi}(q, \eta) = \frac{1}{\eta} \psi_\mu^{(1)}(q) + \theta_\mu^{(1)}(q, \eta) \tag{5.14a}$$

$$\nabla_\mu \nabla_\nu \bar{\phi}(q, \eta) = \frac{1}{2} \psi_{\mu\nu}^{(2)}(q) + \theta_{\mu\nu}^{(2)}(q, \eta) \tag{5.14b}$$

with

$$\lim_{\eta \to 0} \eta \theta_\mu^{(1)}(q, \eta) = 0 \tag{5.14c}$$

$$\lim_{\eta \to 0} \eta \theta_{\mu\nu}^{(2)}(q, \eta) = 0 \tag{5.14d}$$

which leads

$$R^\mu(q, \eta \to 0) = h^\mu + \tfrac{1}{2}\hat{Q}^{\mu\nu}\psi_\nu^{(1)} \tag{5.15}$$

$$\nabla_\mu R^\mu(q, \eta \to 0) = \nabla_\mu h^\mu + \tfrac{1}{2}\hat{Q}^{\mu\nu}\psi_{\mu\nu}^{(2)} \tag{5.16}$$

i.e., still well-defined finite expressions for R^μ, $\nabla_\mu R^\mu$. If we assume the form (5.14), $\psi_\mu^{(1)}$ will satisfy

$$\tfrac{1}{2}\hat{Q}^{\mu\nu}\psi_\mu^{(1)}\psi_\nu^{(1)} + h^\nu\psi_\nu^{(1)} = 0 \tag{5.17}$$

which is similar to the Hamilton–Jacobi equation (5.4a) but $\psi_\mu^{(1)}$ is not the gradient of any function $\bar{\varphi}$.

It is easy to construct models with a $\bar{\varphi}$ having all the regularity properties and satisfying the Hamilton–Jacobi equation (5.4a), but with $\nabla_\mu R^\mu \neq 0$, i.e., with

$$\bar{\varphi}(q) \neq \bar{\phi}(q, \eta = 1) \tag{5.18}$$

$$\bar{\phi}(q, \eta) = \frac{1}{\eta}\bar{\varphi}(q) + \theta(q, \eta) \tag{5.19}$$

We can use the ideas of Section 3, Eqs. (3.2)–(3.5) with a slight modification. We take a $Q^{\mu\nu}$ and a $\bar{\varphi}$ satisfying the regularity and boundary conditions and define

$$\tilde{D}^\mu = \tfrac{1}{2}Q^{\mu\nu}\nabla_\nu\bar{\varphi} \tag{5.20}$$

$$\tilde{r}^\mu = -\tfrac{1}{2}\tilde{d}^{\mu\nu}\nabla_\nu\bar{\varphi} \tag{5.21}$$

where we have chosen any antisymmetric $\tilde{d}^{\mu\nu}$ such that

$$\nabla_\nu\tilde{d}^{\nu\mu}\nabla_\mu\bar{\varphi} \neq 0 \tag{5.22}$$

which is easy to find. Then $\bar{\varphi}$ and $\tilde{r}^\mu$ will satisfy

$$\nabla_\mu\bar{\varphi}\,\tilde{r}^\mu = 0 \tag{5.23}$$

but

$$\nabla_\mu\tilde{r}^\mu \neq 0 \tag{5.24}$$

i.e., if we define a drift

$$h^\mu = \tilde{r}^\mu + \tilde{D}^\mu \tag{5.25}$$

then $\bar{\varphi}(q)$ will satisfy the Hamilton–Jacobi equation (5.4a), but it will not satisfy the Fokker–Planck equation because of (5.24). We finally remark that the exact $\bar{\phi}(q, \eta)$ always satisfies an equation of Hamilton–Jacobi type, as one sees by multiplying (3.1a) by $\nabla_\mu \bar{\phi}$:

$$\tfrac{1}{2} Q^{\mu\nu} \nabla_\mu \bar{\phi} \nabla_\nu \bar{\phi} + [h^\mu - R^\mu] \nabla_\mu \bar{\phi} = 0 \tag{5.26}$$

5.2. Large-Noise Asymptotics

For the large-noise asymptotics $\eta \to \infty$ we make the following ansatz, suggested by the discussion of Eq. (5.8):

$$\bar{\phi}(q, \eta) = \frac{1}{\eta} \bar{\varphi}(q) + O(q, \eta) \tag{5.27}$$

Assuming differentiability and

$$\lim_{\eta \to \infty} \frac{1}{\eta} \theta(q, \eta) = 0 \tag{5.28a}$$

$$\lim_{\eta \to \infty} \frac{1}{\eta} \nabla_\nu \theta(q, \eta) = 0 \tag{5.28b}$$

$$\lim_{\eta \to \infty} \frac{1}{\eta} \nabla_\nu \nabla_\mu \theta(q, \eta) = 0 \tag{5.28c}$$

we get in leading order the Poisson equation

$$\tfrac{1}{2} Q^{\mu\nu} \nabla_\mu \nabla_\nu \bar{\varphi} = -\nabla_\mu h^\mu \tag{5.29}$$

With this ansatz the large-noise asymptotics is better behaved than the small-noise one since the Poisson equation has a smooth solution in the general case.

6. TIME-DEPENDENT PROPERTIES

In this section we will discuss how R^μ affects the time-dependent behavior of the system. We write the covariant Fokker–Planck equation (1.5) as

$$\partial_t S_t(q) = L_F(q)\, S_t(q) \tag{6.1}$$

where $L_F(q)$ is the operator defined by

$$L_F(q)\, f(q) = -\nabla_\mu(h^\mu(q)\, f(q)) + \tfrac{1}{2}Q^{\mu\nu}(q)\, \nabla_\mu \nabla_\nu f(q) \tag{6.2}$$

L_F will be always applied on scalar functions. We define an associated operator, using the stationary $\bar{\phi}$

$$L \equiv e^{(1/2)\bar{\phi}} L_F e^{-(1/2)\bar{\phi}} \tag{6.3}$$

We assume that L has a purely discrete spectrum, and write S_t formally in an eigenvunction expansion[6]

$$S_t(q) = \psi_0(q) \sum_i e^{-\lambda_i t} \psi_i(q) \tag{6.4}$$

where ψ_i, λ_i are the eigenfunctions and eigenvalues of L. It can be shown[6] that (when a stationary state exists) the real part of the λ_i is always positive. In general L will not be hermitic, but it can be decomposed in an hermitic and an anti-hermitic part

$$L = L_H + L_A \tag{6.5}$$

which are given by[6,7]

$$L_H = \tfrac{1}{2}Q^{\mu\nu}(q)\, \nabla_\mu \nabla_\nu - V(q) \tag{6.6}$$

$$L_A = -\tfrac{1}{2}\nabla_\mu R^\mu - R^\mu \nabla_\mu \tag{6.7}$$

with

$$\begin{aligned} V(q) &= -\tfrac{1}{4}Q^{\mu\nu}\, \nabla_\mu \nabla_\nu \bar{\phi} + \tfrac{1}{8}Q^{\mu\nu}\, \nabla_\mu \bar{\phi}\, \nabla_\nu \bar{\phi} \\ &= \tfrac{1}{2}Q_{\mu\nu}\, \bar{D}^\mu \bar{D}^\nu + \tfrac{1}{2}\nabla_\mu \bar{D}^\mu \end{aligned} \tag{6.8}$$

L_A is determined by R^μ and L_H by $\bar{D}^\mu$ and $Q^{\mu\nu}$. We consider two special cases:

(1) $L_A = 0$; i.e., L is hermitic. Then all the eigenvalues λ_i are real and positive and therefore the dynamics is purely diffusive. The form (6.7) implies[6,7] that this is the case if and only if $R^\mu = 0$, or equivalently, if the stationary state is in detailed balance db(1) [as follows from the discussion of Eq. (4.9)].

(2) The next simple case is when $L_A \neq 0$ but $[L_H, L_A] = 0$. Then L is a normal operator ($[L, L^+] = 0$, which allows, e.g., to use the spectral decomposition). The dynamics has a diffusive and a convective part but they do not interfere with each other. We can show that the commutativity of L_H and L_A depends only on R^μ and $Q^{\mu\nu}$ (i.e., not on $\bar{D}^\mu$ or $\bar{\phi}$).

A necessary and sufficient condition for $[L_H, L_A]=0$ is that the tensor defined by $Q^{\mu\nu}\nabla_\nu R^\lambda$ is antisymmetric

$$Q^{\mu\nu}\nabla_\nu R^\lambda = -Q^{\lambda\nu}\nabla_\nu R^\mu \tag{6.9}$$

and that

$$M_{\mu\nu}R^\mu = 0 \tag{6.10}$$

where $M_{\mu\nu}$ is the Ricci tensor associated with the metric $Q^{\mu\nu}(q)$.[24]

The proof of this statement is given in Ref. 25. We remark that for a flat metric (i.e., $Q^{\mu\nu}$ is constant in some coordinate system), (6.10) is automatically satisfied. In the special case $Q^{\mu\nu}=\delta^{\mu\nu}$, the condition (6.9) implies $\nabla_\mu R^\mu = 0$ (but not the other way around). In general the conditions (6.9) and $\nabla_\mu R^\mu = 0$ are independent of each other, in the sense that there are systems satisfying one of them but not the other.

Suppose that we construct a model as described in Section 3 by giving $\bar{\phi}$ and $Q^{\mu\nu}$ and setting $R^\mu=0$. The time evolution is determined by the real positive eigenvalues λ_j^H and eigenfunctions ψ_j. It will be a purely diffusive relaxation. If one then adds a term R^μ to the drift such that (6.9) and (6.10) are satisfied, the dynamics will be determined by $L=L_H+L_A$ with $[L_H, L_A]=0$. L_H and L_A will have the eigenfunctions in common and the eigenvalues of L will be $\lambda_j=\lambda_j^H+\lambda_j^A$, with purely imaginary λ_j^A. The eigenfunctions and the real part of the eigenvalues will not be changed. The convective behavior induced by R^μ will be superposed to the relaxation without modifying it; they are independent from each other.

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